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An isochronous variant of the Ruijsenaars–Toda model: equilibrium configurations, behavior in their neighborhood, Diophantine relations

R Droghei[1,2], **F Calogero**[3,4] **and O Ragnisco**[1,2]

[1] Dipartimento di Fisica, Università Roma Tre, Via Vasca Navale 84, I-00146 Roma
[2] Istituto Nazionale di Fisica Nucleare, Sezione di Roma Tre, Italy
[3] Dipartimento di Fisica, Università di Roma 'La Sapienza', P.le A.Moro 2, I-00185 Roma
[4] Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy

E-mail: droghei@fis.uniroma3.it, francesco.calogero@roma1.infn.it and ragnisco@fis.uniroma3.it

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Abstract
An *isochronous* variant of the Ruijsenaars–Toda *integrable* many-body problem is introduced, an equilibrium configuration of this dynamical system is identified and by investigating the motions in its neighborhood *Diophantine* relations are obtained.

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1. Introduction and main results

Some years ago a technique was introduced [1] which is suitable to modify certain dynamical systems so that the modified systems thereby obtained are *isochronous*, featuring in their phase space a fully dimensional region where *all* their solutions are *isochronous*, i.e. *periodic* in *all* their degrees of freedom with a *fixed* period (independent of the initial data provided they belong to the *isochrony* region; for a review of these findings see the recent monograph [2]). If the original system is *integrable*, there are reasons [2] to expect that the region of *isochrony* of the modified, *isochronous* system coincides with the *entire* phase space. It is then clear that, if such an *isochronous* system possesses an equilibrium configuration, the standard *linearization* of the equations of motion in its (infinitesimal) neighborhood yields a matrix featuring a *Diophantine* property: indeed, since the eigenvalues of this matrix provide the periods of the oscillations of the system in that neighborhood, and since this motion must be *completely periodic* with a *fixed* period, *all* these eigenvalues must be *integer* multiples of a known number. This technique has been applied repeatedly, taking as point of departure various *integrable* systems (for a review see [2]). Its applicability requires two minor miracles: firstly, the possibility of transforming the original, *integrable* system into an *isochronous* system; and secondly, the *explicit* identification of an equilibrium configuration, allowing one to obtain—via a standard procedure, starting from the equations of motion of the *isochronous* system—an *explicit* matrix characterizing the *linearized* equations describing the motion near equilibrium. The outcome of this procedure is the *explicit* identification of a matrix with the *Diophantine* property that *all* its eigenvalues are *integer* numbers. The order of this matrix generally coincides with the number of dependent variables of the system that often can be *arbitrarily large*. When attention is restricted to *few* dependent variables—hence the order of the relevant matrix is *small*—the *Diophantine* property can be verified by computing explicitly the eigenvalues of this matrix; in this manner it is also generally possible to *conjecture* the specific values of these eigenvalues when the order of the matrix is instead *arbitrary*. In some cases these *conjectures* have then been proven (see the monograph [2] and references quoted there; indeed the goal to prove these conjectures has motivated a research development involving *orthogonal polynomials* and *discrete integrable systems* [3–6]).

This paper provides one more example in which the two minor miracles mentioned above do happen. The treatment is analogous to that of two previous papers [2, 7, 8], but the conjectures generally yielded by this approach are in this case replaced by proven statements. Our main results are the identification of the *isochronous* variant of the Ruijsenaars–Toda model [9] (also called 'relativistic Toda', and hereafter referred to by the acronym RT), as described in the following section, see (19), and the identification of the following two tridiagonal $M \times M$ matrices $\underline{A}^{(M)}$ and $\underline{B}^{(M)}$ defined componentwise as follows:

$$A_{1,1}^{(M)} = -M, \qquad A_{1,2}^{(M)} = -(M-1), \tag{1a}$$

$$A_{\ell,\ell-1}^{(M)} = -(\ell-1-M), \qquad A_{\ell,\ell+1}^{(M)} = (\ell-M), \qquad \ell = 2, \ldots, M, \tag{1b}$$

$$B_{1,2}^{(M)} = M(M-1), \tag{2a}$$

$$B_{2,2}^{(M)} = -2(M-1)^2, \qquad B_{2,3}^{(M)} = (M-1)(M-2), \tag{2b}$$

$$\begin{aligned} &B_{\ell,\ell-1}^{(M)} = (\ell-1-M)(\ell-2-M), \qquad B_{\ell,\ell}^{(M)} = -2(\ell-1-M)^2, \\ &B_{\ell,\ell+1}^{(M)} = (\ell-M)(\ell-1-M), \qquad \ell = 3, \ldots, M, \end{aligned} \tag{2c}$$

with all their other matrix elements (i.e. all those not displayed above) vanishing, and the statement that the $2M$ roots of the polynomial $\det[x^2\underline{1}^{(M)} + x\underline{A}^{(M)} + \underline{B}^{(M)}]$, of degree $N = 2M$ in x, are *all integers*, indeed that they are identified by the factorization

$$\det[x^2\underline{1}^{(M)} + x\underline{A}^{(M)} + \underline{B}^{(M)}] = \left[\prod_{m=1}^{M}(x-m)\right]\left[\prod_{m=0}^{M-1}(x+m)\right] = x(x-M)\prod_{m=1}^{M-1}(x^2-m^2). \tag{3}$$

Here and throughout M is an arbitrary integer (larger than 2, $M > 2$, to avoid having to detail separately trivial special cases), and of course $\underline{1}^{(M)}$ denotes the *unit* matrix of order M.

A related finding—instrumental to prove result (3), but of interest in its own right—is obtained by introducing the polynomials $p_n^{(\nu)}(x)$, of *even* degree $n = 2m$ in x and depending on an arbitrary parameter ν, defined by the formula

$$p_n^{(\nu)}(x) = \det[x^2\underline{1}^{(m)} + x\underline{A}^{(m,\nu)} + \underline{B}^{(m,\nu)}], \qquad n = 2m, \tag{4}$$

where the two tridiagonal $m \times m$ matrices $\underline{A}^{(m,\nu)}$ and $\underline{B}^{(m,\nu)}$ are defined (componentwise) as follows:

$$A_{1,1}^{(m,\nu)} = -\nu, \qquad A_{1,2}^{(m,\nu)} = -(\nu-1), \tag{5a}$$

$$A^{(m,\nu)}_{\ell,\ell-1} = -(\ell - 1 - \nu), \qquad A^{(m,\nu)}_{\ell,\ell+1} = (\ell - \nu), \qquad \ell = 2, \ldots, m, \tag{5b}$$

$$B^{(m,\nu)}_{1,2} = \nu(\nu - 1), \tag{6a}$$

$$B^{(m,\nu)}_{2,2} = -2(\nu - 1)^2, \qquad B^{(m,\nu)}_{2,3} = (\nu - 1)(\nu - 2), \tag{6b}$$

$$\begin{aligned} &B^{(m,\nu)}_{\ell,\ell-1} = (\ell - 1 - \nu)(\ell - 2 - \nu), \qquad B^{(m,\nu)}_{\ell,\ell} = -2(\ell - 1 - \nu)^2, \\ &B^{(m,\nu)}_{\ell,\ell+1} = (\ell - \nu)(\ell - 1 - \nu), \qquad \ell = 3, \ldots, m, \end{aligned} \tag{6c}$$

with all their other matrix elements vanishing (and m an arbitrary integer larger than 2, $m > 2$). As noted in the following section, these polynomials $p^{(\nu)}_n(x)$ (of *even* degree $n = 2m$) can also be defined via the three-term recursion relation

$$\begin{aligned} p^{(\nu)}_{2(m+1)}(x) &= [x^2 - 2(m - \nu)^2]p^{(\nu)}_{2m}(x) + (m - \nu)^2[x^2 - (m - 1 - \nu)^2]p^{(\nu)}_{2(m-1)}(x), \\ &m = 2, 3, \ldots, \end{aligned} \tag{7a}$$

with the initial assignments

$$p^{(\nu)}_2(x) = x(x - \nu), \tag{7b}$$

$$p^{(\nu)}_4(x) = x(x - \nu)[x^2 - (1 - \nu)^2]. \tag{7c}$$

And it is then clear that the *Diophantine* factorization holds:

$$\begin{aligned} p^{(m)}_{2m}(x) &= \left[\prod_{\ell=1}^{m}(x - \ell)\right]\left[\prod_{\ell=0}^{m-1}(x + \ell)\right] \\ &= x(x - m)\prod_{\ell=1}^{m-1}(x^2 - \ell^2), \end{aligned} \tag{8}$$

as implied by formulas (3) and (4), together with the observation that clearly

$$\underline{A}^{(M,M)} = \underline{A}^{(M)}, \qquad \underline{B}^{(M,M)} = \underline{B}^{(M)}, \tag{9}$$

as implied by a comparison of (1) with (5) and of (2) with (6). But—as shown in the following section—there holds in fact the following more general result:

$$p^{(\nu)}_{2m}(x) = x(x - \nu)\prod_{\ell=1}^{m-1}[x^2 - (\ell - \nu)^2], \tag{10}$$

which clearly reduces to (8) for $\nu = m$.

This finding suggests introducing a second family of polynomials $q^{(\nu)}_m(z)$, of degree m in z, via the *ansatz*

$$p^{(\nu)}_{2m}(x) = x(x - \nu)q^{(\nu)}_{m-1}(z), \qquad z = x^2, \tag{11}$$

so that also these polynomials are trivially factorized,

$$q^{(\nu)}_m(z) = \prod_{\ell=1}^{m}[z - (\ell - \nu)^2]. \tag{12}$$

Moreover, these polynomials satisfy the three-term recursion relation

$$q^{(\nu)}_m(z) = [z - 2(m - \nu)^2]q^{(\nu)}_{m-1}(z) + (m - \nu)^2[z - (m - 1 - \nu)^2]q^{(\nu)}_{m-2}(z), \tag{13a}$$

being indeed also *defined* (for all nonnegative integer values of their order m) by these recursion relations together with the initial assignments

$$q^{(\nu)}_0(z) = 1, \qquad q^{(\nu)}_1(z) = z - (1 - \nu)^2, \tag{13b}$$

and they also satisfy the second recursion relation (for whose relevance see [3–6])

$$q_m^{(\nu+1)}(z) = q_m^{(\nu)}(z) + g_m^{(\nu)} q_{m-1}^{(\nu)}(z), \qquad m = 1, 2, \ldots, \tag{14a}$$

with

$$g_m^{(\nu)} = m(m - 2\nu). \tag{14b}$$

The derivation of all these findings is detailed in the following section.

2. Derivation of the results

Our point of departure is the RT *integrable* dynamical system [9] characterized by the Hamiltonian

$$\mathcal{H} = \sum_{m=1}^{M} \left\{ \exp(p_m)[1 + \exp(q_{m-1} - q_m)]^{\frac{1}{2}} [1 + \exp(q_m - q_{m+1})]^{\frac{1}{2}} - 2 \right\}, \tag{15}$$

whose equations of motion are, for our purposes, conveniently written as follows [10, 11]:

$$a_m' = a_m(1 - a_m)(b_m - b_{m+1}), \tag{16a}$$

$$b_m' = b_m(b_{m-1}a_{m-1} - b_{m+1}a_m). \tag{16b}$$

Notation 1. *Here $a_m \equiv a_m(\tau)$ and $b_m \equiv b_m(\tau)$ are the dependent variables and are related to the canonical coordinates $q_m(\tau)$, $p_m(\tau)$ as follows:*

$$a_m = \frac{\exp(q_m - q_{m+1})}{[1 + \exp(q_m - q_{m+1})]}, \tag{17a}$$

$$b_m = \exp(p_m)[1 + \exp(q_{m-1} - q_m)]^{\frac{1}{2}} [1 + \exp(q_m - q_{m+1})]^{\frac{1}{2}}; \tag{17b}$$

while τ is of course the independent variable, and appended primes denote differentiation with respect to τ. Here and hereafter (unless otherwise indicated) the index m runs from 1 to M, with M being an arbitrary positive integer (but occasionally we shall understand that $M > 2$, to avoid having to single out trivial cases).

This dynamical system is complemented by the boundary conditions

$$a_0 = a_M = 0, \qquad b_0 = b_{M+1} = 0 \tag{17c}$$

(corresponding to $q_0 = -\infty, q_{M+1} = +\infty$ and $p_0 = p_{M+1} = -\infty$, see (17)), which are known to be compatible with its integrability [10]. Note that this system features formally $2M$-dependent variables but has in fact only $2M - 1$ nontrivial-dependent variables, since the variable a_M vanishes identically (see (17c), and note the compatibility of this condition with (16a)).

This system is transformed into an *isochronous* system by introducing new dependent and independent variables via the following assignment [2]:

$$\alpha_m(t) = a_m(\tau), \qquad \beta_m(t) = \exp(\mathrm{i}t) b_m(\tau) \tag{18a}$$

with

$$\tau = -\mathrm{i}[\exp(\mathrm{i}t) - 1] \tag{18b}$$

entailing (for simplicity) $\tau(0) = 0$; hence $\alpha_m(0) = a_m(0)$, $\beta_m(0) = b_m(0)$. The equations of motion of the *isochronous* system then clearly read

$$\dot{\alpha}_m = \alpha_m(1 - \alpha_m)(\beta_m - \beta_{m+1}), \tag{19a}$$

$$\dot{\beta}_m - \mathrm{i}\beta_m = \beta_m(\alpha_{m-1}\beta_{m-1} - \alpha_m\beta_{m+1}), \tag{19b}$$

and are complemented by the boundary conditions

$$\alpha_0 = \alpha_M = 0, \qquad \beta_0 = \beta_{M+1} = 0. \tag{19c}$$

Notation 2. *Here and hereafter the superimposed dot indicates of course differentiation with respect to the new (real) independent variable t ('time'), and* i *is the imaginary unit,* $\mathrm{i}^2 = -1$.

The *autonomous* character of this system (19) is the first of the two minor miracles mentioned above.

As implied by the transformation (18)—together with the (Painlevé) property of the solutions of the original dynamical system (16) to be *meromorphic* functions of their independent variable τ —the property of *isochrony* of this system (19) reads

$$\alpha_m(t + 2\pi) = \alpha_m(t), \qquad \beta_m(t + 2\pi) = \beta_m(t). \tag{20}$$

Next, let us consider the equilibrium configurations of this dynamical system, (19). They are clearly characterized by the values $\bar{\alpha}_m, \bar{\beta}_m$ of the dependent variables satisfying the system of $N = 2M$ algebraic equations

$$\bar{\alpha}_m(1 - \bar{\alpha}_m)(\bar{\beta}_m - \bar{\beta}_{m+1}) = 0, \tag{21a}$$

$$-\mathrm{i}\bar{\beta}_m = \bar{\beta}_m(\bar{\alpha}_{m-1}\bar{\beta}_{m-1} - \bar{\alpha}_m\bar{\beta}_{m+1}), \tag{21b}$$

complemented of course by the boundary conditions

$$\bar{\alpha}_0 = \bar{\alpha}_M = 0, \qquad \bar{\beta}_0 = \bar{\beta}_{M+1} = 0. \tag{21c}$$

As can be easily verified (separately for $m = 1$, for $m = 2$, for $m = 3, \ldots, M - 1$ and for $m = M$), a solution of this system of algebraic equations (21) reads

$$\bar{\alpha}_1 = 1, \qquad \bar{\beta}_1 = -(M - 1)\mathrm{i}, \tag{22a}$$

$$\bar{\alpha}_m = -(M - m), \qquad \bar{\beta}_m = \mathrm{i}, \qquad m = 2, \ldots, M \tag{22b}$$

(note the consistency of (22*b*), for $m = M$, with (21*c*)).

This is not the only equilibrium configuration of the dynamical system (19), but it seems to be the only one useful for our purposes. And the fact that this equilibrium configuration is *explicit* is the second of the two minor miracles mentioned above.

Next, let us obtain the linearized system that characterizes the behavior of the *isochronous* dynamical system (19) in the immediate neighborhood of this equilibrium configuration, (21). This obtains, in the limit when ε is *infinitesimal*, by setting

$$\alpha_m(t) = \bar{\alpha}_m + \varepsilon x_m(t), \qquad \beta_m(t) = \bar{\beta}_m + \varepsilon y_m(t). \tag{23}$$

Note that the *isochronous* character of $\alpha_m(t)$ and $\beta_m(t)$, see (20), entails the analogous property for $x_m(t)$ and $y_m(t)$:

$$x_m(t + 2\pi) = x_m(t), \qquad y_m(t + 2\pi) = y_m(t). \tag{24}$$

The insertion of this *ansatz* (23) in the equations of motion (19) yields the *linear* system

$$\dot{x}_m = (1 - 2\bar{\alpha}_m)(\bar{\beta}_m - \bar{\beta}_{m+1})x_m + \bar{\alpha}_m(1 - \bar{\alpha}_m)(y_m - y_{m+1}), \tag{25a}$$

$$\begin{aligned}\dot{y}_m - \mathrm{i}y_m = {} & \bar{\beta}_m(\bar{\beta}_{m-1}x_{m-1} - \bar{\beta}_{m+1}x_m + \bar{\alpha}_{m-1}y_{m-1} - \bar{\alpha}_m y_{m+1}) \\ & + (\bar{\alpha}_{m-1}\bar{\beta}_{m-1} - \bar{\alpha}_m\bar{\beta}_{m+1})y_m,\end{aligned} \tag{25b}$$

complemented of course by the boundary conditions (see (19*c*) and (21*c*))

$$x_0 = x_M = 0, \qquad y_0 = y_{M+1} = 0. \tag{25c}$$

And the insertion of the equilibrium data (22) in this system of ODEs yields the following system of $N = 2M - 1$ linear first-order ODEs:

$$\dot{x}_1 = M\mathrm{i}x_1, \tag{26a}$$

$$\dot{x}_\ell = -(M-\ell)(M-\ell+1)(y_\ell - y_{\ell+1}), \qquad \ell = 2, \ldots, M-1, \tag{26b}$$

$$\dot{y}_1 = -(M-1)x_1 + (M-1)\mathrm{i}y_2, \tag{26c}$$

$$\dot{y}_2 = (M-1)x_1 + x_2 + \mathrm{i}y_1 + (M-2)\mathrm{i}y_3, \tag{26d}$$

$$\dot{y}_\ell = -x_{\ell-1} + x_\ell - (M-\ell+1)\mathrm{i}y_{\ell-1} + (M-\ell)\mathrm{i}y_{\ell+1}, \qquad \ell = 3, \ldots, M, \tag{26e}$$

$$\dot{y}_M = -x_{M-1} - \mathrm{i}y_{M-1}. \tag{26f}$$

Note that the last formula (26*f*) coincides, via (25*c*), with the next-to-last formula (26*e*) with $\ell = M$ (hence, this is again a system of $2M - 1$ first-order ODEs).

This system can clearly be reformulated as a system of M linear *second-order* ODEs for the M-dependent variables, y_m, reading

$$\ddot{y}_1 - \mathrm{i}M\dot{y}_1 - \mathrm{i}(M-1)\dot{y}_2 - M(M-1)y_2 = 0, \tag{27}$$

$$\ddot{y}_2 + \mathrm{i}(M-1)\dot{y}_1 - \mathrm{i}(M-2)\dot{y}_3 + 2(M-1)^2 y_2 - (M-1)(M-2)y_3 = 0, \tag{28}$$

$$\begin{aligned}\ddot{y}_\ell &+ \mathrm{i}(M-\ell+1)\dot{y}_{\ell-1} - \mathrm{i}(M-\ell)\dot{y}_{\ell+1} - (M-\ell+1)(M-\ell+2)y_{\ell-1}\\ &+ 2(M-\ell+1)^2 y_\ell - (M-\ell)(M-\ell+1)y_{\ell+1} = 0, \quad \ell = 3, \ldots, M,\end{aligned} \tag{29}$$

namely

$$\ddot{\underline{y}} + \mathrm{i}\underline{A}^{(M)}\dot{\underline{y}} - \underline{B}^{(M)}\underline{y} = 0, \tag{30}$$

where the two (constant) $M \times M$ matrices $\underline{A}^{(M)}$ and $\underline{B}^{(M)}$ are defined above, see (1) and (2), and of course the M-vector $\underline{y} \equiv \underline{y}(t)$ has the M components $y_m \equiv y_m(t)$.

The general solution of this system of *linear* ODEs (30) is provided by the formula

$$\underline{y}(t) = \sum_{k=1}^{2M} c_k \exp(\mathrm{i}x_k t)\underline{u}^{(k)}, \tag{31}$$

where the $2M$ numbers x_k are the $2M$ roots of the polynomial $\det[x^2\underline{1}^{(M)} + x\underline{A}^{(M)} + \underline{B}^{(M)}]$, of degree $2M$ in x, the $2M$ constants c_k are arbitrary and the $2M$ constant M-vectors are defined appropriately.

The *isochronous* character, see (24), of this (general) solution of the system (30) entails the *Diophantine* conclusion that the $2M$ numbers x_k must *all* be *integers*.

The neater way to verify this result, and to moreover identify that these $2M$ integer numbers x_k are those implied by the right-hand side of the explicit factorization formula (3),

$$x_m = m \quad \text{for} \quad m = 1, \ldots, M, \qquad x_{m+M+1} = -m \quad \text{for} \quad m = 0, \ldots, M-1, \tag{32}$$

is to prove all the results reported in the previous section backwards, starting from the last one reported there (while the order in which they are reported above reflects more faithfully the sequential character of their original derivation, via theorems and conjectures analogous to those of previous treatments [3, 7, 8]).

Let us therefore define the polynomials $q_m^{(\nu)}(z)$ via the explicit factorization formula (12). It is then trivial to verify, by direct substitution, that these polynomials satisfy the two recursion relations (13) and (14). Next, we define the polynomials $p_n^{(\nu)}(x)$ via (11), so that they feature the factorization (10); it is then trivial to also verify, again by direct substitution, that they satisfy the recursion relation (7). The fact that (4) (with (5) and (6)) is consistent with these three-term recursion relations (7) is then easily verified by evaluating the determinant of the *tridiagonal* matrix $\underline{C}^{(m,\nu)}(x) \equiv x^2\underline{1}^{(m)} + x\underline{A}^{(m,\nu)} + \underline{B}^{(m,\nu)}$, by expanding it according to the formula

$$\begin{aligned}\det[\underline{C}^{(m,\nu)}(x)] &= C_{m,m}^{(m,\nu)}\det[\underline{C}^{(m-1,\nu)}(x)] - C_{m,m-1}^{(m,\nu)}C_{m-1,m}^{(m,\nu)}\det[\underline{C}^{(m-2,\nu)}(x)]\\ &= \left(x^2 + B_{m,m}^{(m,\nu)}\right)\det[\underline{C}^{(m-1,\nu)}(x)]\\ &\quad - \left(xA_{m,m-1}^{(m,\nu)} + B_{m,m-1}^{(m,\nu)}\right)\left(xA_{m-1,m}^{(m,\nu)} + B_{m-1,m}^{(m,\nu)}\right)\det[\underline{C}^{(m-2,\nu)}(x)],\end{aligned} \tag{33}$$

entailed by the tridiagonal character of the matrices $\underline{A}^{(m,\nu)}$ and $\underline{B}^{(m,\nu)}$, by the fact that $A_{m,m}^{(m,\nu)}$ vanishes (see (5)) and, most importantly, by the fact that the matrix elements of these two tridiagonal matrices are independent of their order m (see (5) and (6)); and then by using (4) to identify this formula (33) with the recursion relation (7) (using the explicit expressions (5) and (6)).

This completes the proof of the results reported in the previous section, since, as already noted there, the factorization (3), or equivalently (8), is evidently a special case of the more general factorization (10).

We complete this paper with two remarks.

Remark 1. Of course the factorization (12) entails that the polynomials $q_m^{(\nu)}(z)$ satisfy the two-term recursion relation

$$q_m^{(\nu)}(z) = [z - (m-\nu)^2]q_m^{(\nu)}(z), \tag{34a}$$

which is easily seen to be compatible—indeed, to imply—the three-term recursion relation (13*a*). And likewise the polynomials $p_{2m}^{(\nu)}(x)$ satisfy the two-term recursion

$$p_{2m}^{(\nu)}(x) = [x^2 - (m-1-\nu)^2]p_{2(m-1)}^{(\nu)}(x), \tag{35}$$

in addition to the three-term recursion (7*a*).

Remark 2. Already in the original paper by Ruijsenaars [9] it was observed that the Poincaré invariance of the RT model entails the existence of two commuting Hamiltonians, $\mathcal{H}_+ \equiv \mathcal{H}$ and $\mathcal{H}_-$—one being the 'time-reversed' variant of the other—related by the involutive symmetry

$$q_m \to q_m, \qquad p_m \to -p_m. \tag{36}$$

Hence, these two Hamiltonians read (compare with (15))

$$\mathcal{H}_\pm = \sum_{n=1}^{M}\left\{\exp(\pm p_m)[1+\exp(q_{m-1}-q_m)]^{\frac{1}{2}}[1+\exp(q_m-q_{m+1})]^{\frac{1}{2}} - 2\right\}. \tag{37}$$

In terms of the variables a_m and b_m related to the canonical variables p_m and q_m by the relations (17), the two Hamiltonian flows associated with $\mathcal{H}_+$ respectively $\mathcal{H}_-$ read

$$\begin{pmatrix}[a_m, \mathcal{H}_+]\\ [b_m, \mathcal{H}_+]\end{pmatrix} = \begin{pmatrix}a_m(1-a_m)(b_m - b_{m+1})\\ b_m(b_{m-1}a_{m-1} - a_m b_{m+1})\end{pmatrix}, \tag{38a}$$

respectively

$$\begin{pmatrix}[a_m, \mathcal{H}_-] \\ [b_m, \mathcal{H}_-]\end{pmatrix} = \begin{pmatrix}\frac{a_m}{b_m(1-a_{m-1})} - \frac{a_m}{b_{m+1}(1-a_{m+1})} \\ \frac{a_m}{1-a_m} - \frac{a_{m-1}}{1-a_{m-1}}\end{pmatrix}, \tag{38b}$$

where the notation $[A, B]$ denotes of course the standard Poisson bracket.

Note that these two Hamiltonian flows can be related by the transformation (36) which, once rewritten in terms of the variables a_m and b_m, reads

$$a_m \to a_m, \qquad b_m \to \frac{1}{b_m(1-a_m)(1-a_{m-1})}. \tag{39}$$

And let us also mention that it has been shown [12] that the two flows (38) are just the first ones of two commuting hierarchies of flows, constructed via a recursion operator and its inverse, and having as Hamiltonians the traces of the positive and negative powers of the same Lax matrix.

Of course the first flow yields the equations of motion (16), while the second flow of the RT system, corresponding to the Hamiltonian $\mathcal{H}_-$, yields, for the dependent variables $a_m \equiv a_m(\tau)$ and $b_m \equiv b_m(\tau)$, the equations of motion

$$a_m' = a_m\left[\frac{1}{b_m(1-a_{m-1})} - \frac{1}{b_{m+1}(1-a_{m+1})}\right], \tag{40a}$$

$$b_m' = \frac{a_m}{(1-a_m)} - \frac{a_{m-1}}{(1-a_{m-1})}, \tag{40b}$$

to be again supplemented with the 'boundary conditions' (17*c*). But it is easily seen that the dynamical system characterized by these equations of motion can be reduced to that considered above, see (16), merely by a more convenient choice of variables, namely by introducing the following 'tilded' variables:

$$\tilde{a}_m = \frac{\exp(q_m - q_{m+1})}{[1+\exp(q_m - q_{m+1})]} = a_m, \tag{41a}$$

$$\begin{aligned}\tilde{b}_m &= -\exp(-p_m)[1+\exp(q_{m-1}-q_m)]^{\frac{1}{2}}[1+\exp(q_m-q_{m+1})]^{\frac{1}{2}} \\ &= -\frac{1}{b_m(1-a_m)(1-a_{m-1})}.\end{aligned} \tag{41b}$$

It is indeed easy to verify that the evolution yielded, for the canonical variables $q_m(\tau)$ and $p_m(\tau)$, by the Hamiltonian $\mathcal{H}_-$ entails, for the tilded variables $\tilde{a}_m(\tau)$ and $\tilde{b}_m(\tau)$, just the same equations of motion,

$$\tilde{a}_m' = \tilde{a}_m(1-\tilde{a}_m)(\tilde{b}_m - \tilde{b}_{m+1}), \tag{42a}$$

$$\tilde{b}_m' = \tilde{b}_m(\tilde{b}_{m-1}\tilde{a}_{m-1} - \tilde{b}_{m+1}\tilde{a}_m), \tag{42b}$$

yielded by the Hamiltonian $\mathcal{H}_+$ for the dependent variables $a_m(\tau)$ and $b_m(\tau)$, see (16).

References

[1] Calogero F 1997 A class of integrable Hamiltonian systems whose solutions are (perhaps) all completely periodic *J. Math. Phys.* **38** 5711–9

[2] Calogero F 2008 *Isochronous Systems* (Oxford: Oxford University Press)

[3] Bruschi M, Calogero F and Droghei R 2007 Proof of certain Diophantine conjectures and identification of remarkable classes of orthogonal polynomials *J. Phys. A: Math. Theor.* **40** 3815–29

[4] Bruschi M, Calogero F and Droghei R 2007 Tridiagonal matrices, orthogonal polynomials and Diophantine relations: I. *J. Phys. A: Math. Theor.* **40** 9793–817

[5] Bruschi M, Calogero F and Droghei R 2007 Tridiagonal matrices, orthogonal polynomials and Diophantine relations: II. *J. Phys. A: Math. Theor.* **40** 14759–72

[6] Bruschi M, Calogero F and Droghei R 2009 Additional recursion relations, factorizations and Diophantine properties associated with the polynomials of the Askey scheme *Advances Math. Phys.* **268134** 43

[7] Calogero F, Di Cerbo L and Droghei R 2006 On isochronous Bruschi–Ragnisco–Ruijsenaars–Toda lattices: equilibrium configurations, behavior in their neighborhood, diophantine relations and conjectures *J. Phys. A: Math. Gen.* **39** 313–25

[8] Calogero F, Di Cerbo L and Droghei R 2006 On isochronous Shabat–Yamilov lattices: equilibrium configurations, behavior in their neighborhood, diophantine relations and conjectures *Phys. Lett.* A **355** 262–70

[9] Ruijsenaars S N M 1990 Relativistic Toda systems *Commun. Math. Phys.* **133** 217–47

[10] Bruschi M and Ragnisco O 1988 Recursion operator and Bäcklund transformations for the Ruijsenaars–Toda lattice *Phys. Lett.* A **129** 21–5

[11] Bruschi M and Ragnisco O 1989 Direct and Inverse Spectral problem for the infinite Ruijsenaars–Toda lattice *Inverse Problems* **5** 389–405

[12] Oevel W, Fuchssteiner B, Zhang H and Ragnisco O 1989 Mastersymmetries, angle variables and recursion operator of the relativistic Toda lattice *J. Math. Phys.* **30** 2664–70